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A PLANE CUBIC CREMONA TRANSFORMATION AND ITS INVERSE.*

BY DR. F. M. MORGAN.

1. In the present paper attention is called to a very simple cubic Cremona transformation of the plane in which a fundamental point A_1 coincides with a fundamental point A_2, where by A_i is understood a fundamental point of multiplicity i. Such an example of a Cremona transformation is of interest because in 1901 Segre† discovered that Noether's‡ proof that every Cremona transformation can be expressed in terms of quadratic transformations is not valid when two fundamental points A_i and A_k come into coincidence. A short time later Castelnuovo§ completed the proof so as to apply to this case as well.

It is also shown in this paper how this cubic Cremona transformation with coincident fundamental points gives a simple ruler and compass construction for some well-known higher plane curves.

Zahradink,|| Aubert and Papelier¶ give the cubic transformation

$$x_1 = \frac{xy^2}{x^2 + y^2}, \qquad y_1 = \frac{x^2y}{x^2 + y^2}, \tag{1}$$

and its inverse

$$x = \frac{x_1^2 + y_1^2}{x_1}, \qquad y = \frac{x_1^2 + y_1^2}{y_1}. \tag{2}$$

Zahradink uses these transformations in considering the cubic curves $(x_1^2 + y_1^2)(ay_1 + bx_1) + cx_1y_1 = 0$ and $axy^2 + byx^2 + c(x^2 + y^2) = 0$, while Aubert and Papelier use them in the following exercise.

Exercise. Consider two rectangular axes OX and OY and any point $M(x, y)$ in the plane. (See figure.) Draw MP and MQ perpendicular to OX and OY respectively. Draw PR and QS perpendicular to OM. Through R and S draw parallels to OX and OY respectively, meeting in U. If the coördinates of U are x_1 and y_1, derive the transformations (1) and (2). By means of these transformations find the transforms of a straight line and of a circle passing through the origin.

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* Read before the American Mathematical Society, September 8, 1914.

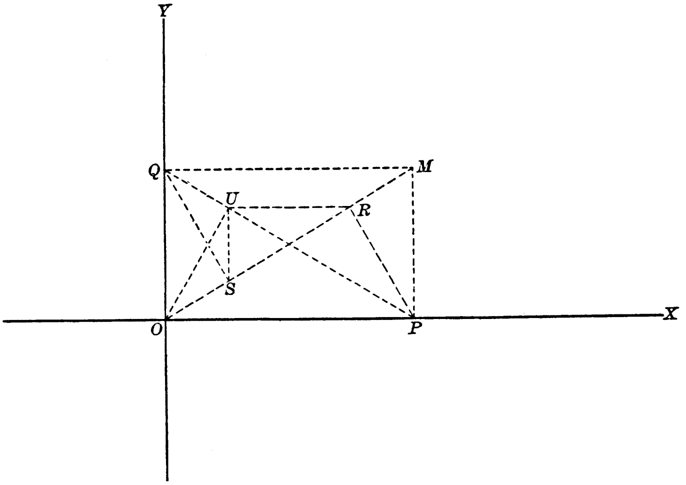
† Atti della Reale Accademia di Torino, vol. 36 (1901), pp. 645–651.

‡ Mathematische Annalen, vol. 3 (1871), p. 164.

§ Atti della Reale Accademia di Torino, vol. 36 (1901), pp. 861–874.

|| Casopis, vol. 34 (1905), pp. 329–340.

¶ Exercises de Géométrie Analytique, vol. 1, pp. 152–153.



The properties of these transformations do not seem to have been discussed further. These may be found much more easily if the transformations are made homogeneous. Calling them S and T, we have

$$S = \left\{ \frac{x}{y_1(x_1^2 + y_1^2)},\ \frac{y}{x_1(x_1^2 + y_1^2)},\ \frac{z}{x_1 y_1 z_1} \right\},$$

$$T = \left\{ \frac{x_1}{xy^2},\ \frac{y_1}{yx^2},\ \frac{z_1}{(x^2 + y^2)z} \right\}.$$

Now if we let

$$I = \left\{ \frac{x_1}{x_2 z_2},\ \frac{y_1}{y_2 z_2},\ \frac{z_1}{x_2^2 + y_2^2} \right\},\quad R = \left\{ \frac{x_2}{yz}\ \frac{y_2}{xz}\ \frac{z_2}{xy} \right\}$$

it follows that $T = RI$. Since however $S^{-1} = T$, we have $S^{-1} = RI$ or $S = I^{-1}R^{-1}$. But R, I, R^{-1}, I^{-1}, are all quadratic transformations, and therefore we have found the quadratic transformations in terms of which S and T may be expressed.

2. The Fundamental System of the Plane S. A cubic Cremona transformation* has $\alpha_2 = 1$, $\alpha_1 = 4$ where α_i means the number of i-fold fundamental points. In any Cremona transformation it is known that the Jacobian curve for the funamental system is identical with the totality of fundamental curves of the net.* The Jacobian in the plane S is $J = x_1 y_1(x_1^2 + y_1^2) = 0$. Thus the fundamental points are $A_2 = (0, 0, 1)$, $A_1^{(1)} = (1, i, 0)$, $A_1^{(2)} = (1, -i, 0)$ and $A_1^{(3)} = (0, 0, 1)$ counted twice. The fundamental curves are four straight lines connecting A_2 to each of the four points A_1, and the conic $x_1^2 + y_1^2 = 0$ which passes through all five fundamental points, (0, 0 1). The point is thus seen to be the union of A_2 and $2A_1$.

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* Doehlemann: Geometrische Transformationen, Part 2, pp. 142–149.

3. The Fundamental System of the Plane T. The Jacobian in the plane T is $J = -3x^2y^2(x^2+y^2) = 0$. The fundamental points are, therefore, $A_2 = (0, 0, 1)$, $A_1^{(1)} = (0, 1, 0)$, $A_1^{(2)} = (1, 0, 0)$ and $A_1^{(3)} = (0, 0, 1)$ counted twice. The fundamental curves are four straight lines connecting A_2 to each of the four points A_1, and the conic $xy = 0$ which passes through all five fundamental points. The point (0, 0, 1) is the union of A_2 and $2A_1$.

4. Geometrical Considerations. *Construction.* A very simple construction in rectangular coördinates for finding the point U, given the point M, is as follows. (See figure.) From M drop perpendiculars MP and MQ to the X and Y axes respectively. Draw the line PQ and drop a perpendicular from O to PQ. The intersection of these two lines is U. This construction has an advantage over that of Aubert and Papelier, in the fact that from this construction it is easy to get M when U is given. For, if we draw UO, erect a perpendicular to it at U and call P and Q the points where it meets the X and Y axes, then OP and OQ are the coördinates of the desired point M.

If the single infinity of curves $x^2 + y^2 = a^2z^2$ is transformed by S, it becomes $(x_1^2 + y_1^2)^3 = a^2x_1^2y_1^2z_1^2$. If this system is first written in non-homogeneous coördinates, and then transformed to polar coördinates it takes the form $2\rho = a \sin 2\theta$ which is the well-known polar equation of a single infinity of four-leaved roses. Therefore using the construction already described, we have a method of constructing a four-leaf rose by means of a ruler and compass.

If on the other hand the system of conics $x_1^2 + y_1^2 = a^2z_1^2$ is transformed by T, the resulting system is $x^2y^2 = a^2(x^2 + y^2)z^2$. In non-homogeneous coördinates the curves of this system are quartic curves with four asymptotes tangent to the circle $x^2 + y^2 = a^2$ and parallel to the axes. The origin is an isolated double point on each curve of the system. The transformation S^2 transforms this system of quartics into the system of roses. T^2 performs the reverse operation.

The system of conics $xy = a^2z^2$ is transformed by S into the system $(x_1^2 + y_1^2)^2 = a^2x_1y_1z_1^2$. This system when reduced to polar coördinates becomes $2\rho^2 = a^2 \sin 2\theta$. Thus from an equilateral hyperbola a lemniscate can be constructed by means of a ruler and compass.

The system of conics $x_1z_1 = a^2y_1^2$ when transformed by S becomes $x(x^2 + y^2) = a^2y^2z^2$ which in non-homogeneous coördinates is at once recognized to be a system of cissoids. Therefore we have a ruler and compass construction for a cissoid.

Thus not only do the transformations (1) and (2) afford examples of Cremona transformations in which two fundamental points coincide, but their geometrical interpretation gives us a ruler and compass construction for some well-known higher plane curves when we start with a conic, a curve constructible by the same process.

DARTMOUTH COLLEGE,
September, 1914.